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A unified approach to discretizations of the linear analysis of thermal regenerators

L. ZHANG and D. M. SCOTT
Department of Chemical Engineering, University of Cambridge, Pembroke Street, Cambridge CB2 3RA, U.K.

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Abstract—Two models of linear thermal regenerator operation are studied using Laplace transform techniques. It is shown that the cell model of Lai, Dudukovic and Ramachandran and the trapezoidal rule discretization of Hill and Willmott are related by changes of time scale and parameters. A matrix formalism is developed for representing transient regenerator operation. The formalism is used to elucidate the physical interpretation of the equations for the cyclic steady state, and to study heat recovery following an impulse perturbation to the gas inlet temperature. The ultimate fractional heat recovery from such a perturbation can be found from the cyclic steady state outlet gas dimensionless temperature distribution; it is bounded by the maximum and minimum values of that distribution.

1. INTRODUCTION

THE THEORY of thermal regenerators has been studied for more than 70 years [1, 2], but over the last decade there has still been considerable interest in numerical methods for solving the basic linear equations for countercurrent regenerator operation [3–6]. The reason is that for computer design of a thermal regenerator for a particular operation, it may be necessary to perform a large number of numerical simulations, and in these circumstances the efficiency of the numerical simulation is very important. An accurate, rapid and robust method has been developed by Hill and Willmott (HW) [5, 6], following the approach proposed by Razelos [3]. Whereas Razelos uses a Euler discretization of the governing equations, Hill and Willmott use a more accurate trapezoidal rule discretization, and also offer improved methods of solving the resulting equations for the cyclic steady state. A slightly different approach was followed by Lai, Dudukovic and Ramachandran (LDR) [4], who set up a cell model of a regenerator which they solved using Laplace transforms. The cell model is a less accurate discretization of the governing differential equations.

An alternative method of thermal regenerator simulation is provided by the Nusselt equations, which lead to integral equations for the cyclic steady state [1, 2]. However, for all practical purposes the integrals must be evaluated numerically, and then, as was pointed out by Burns [7], there is no advantage in using integral equations over partial differential equations.

The present work does not offer a new scheme of calculation. Instead, a comparison is made of the cell model of LDR and the trapezium rule discretization of HW. While, in terms of truncation error, HW is more accurate than LDR, it is shown that the algebraic structure of the two models is identical, and that numerical results from the two models can be related by scaling the dimensionless time variable and by modifying parameters. Further, a formalism is developed which clarifies the algebraic structure of the two models, in particular the origin and form of the closed method equations for the cyclic steady state developed by HW [5]. An example of its use is given, which is relevant to transient regenerator operation. This is a study of the heat recovery following an impulse disturbance to the inlet gas temperature of a regenerator in cyclic steady state. It is shown how the fractions of the heat from the impulse which are eventually lost from each end of the regenerator are related to the cyclic steady state outlet gas temperature distribution.

2. THE LAPLACE TRANSFORM METHOD

The equations governing linear regenerator operation are [1, 2]

$$\frac{\partial \Theta}{\partial \tau} = \frac{hA}{MC}(\theta - \Theta) \tag{1}$$

$$\frac{\partial \theta}{\partial x} = \frac{hA}{WSL}(\Theta - \theta). \tag{2}$$

The usual assumptions have been made that the gas is in plug flow, that there is no radial heat transfer and that the accumulation term in the gas thermal balance can be neglected [1, 2]. Now we define dimensionless solid and gas temperatures

$$T = \frac{\Theta - \theta_c}{\theta_h - \theta_c} \tag{3}$$

NOMENCLATURE

A	total heat transfer surface area [m^2]
C	specific heat of solid [$J\ kg^{-1}$]
h	gas to solid heat transfer coefficient [$W\ m^{-2}\ K^{-1}$]
L	length of regenerator [m]
M	mass of solid packing [kg]
N	maximum value of cell or node label n
Q_m	fraction of heat lost from cold end of regenerator during mth hot period
Q_{tot}	total fraction of heat lost from cold end of regenerator
S	specific heat of gas [$J\ kg^{-1}$]
t_c	dimensionless cold inlet gas temperature
t_h	dimensionless hot inlet gas temperature
t_n	dimensionless gas temperature
t_0	dimensionless inlet gas temperature
$t_N^{(m,h)}$	dimensionless gas outlet temperature during mth hot period
T_n	dimensionless solid temperature
$\mathbf{T}$	vector of dimensionless solid temperatures
$\mathbf{T}^{m,c}$	dimensionless solid temperature at the end of the mth cold period
$\mathbf{T}^{m,h}$	dimensionless solid temperature at the end of the mth hot period
V_n	dimensionless solid temperature
W	gas mass flow rate [$kg\ s^{-1}$]
x	distance coordinate [m].

Greek symbols

α, β, γ	coefficient functions, defined in the Appendix
Δt	variation in outlet gas temperature during a period
ε	thermal effectiveness
θ	gas temperature [K]
θ_c	cold reference gas temperature [K]
θ_h	hot reference gas temperature [K]
Θ	solid temperature [K]
λ	dimensionless length per cell
Λ	dimensionless (reduced) length
η	dimensionless time
η_c	dimensionless cold period
η_h	dimensionless hot period
ξ	dimensionless distance coordinate
Π	dimensionless (reduced) period
τ	time [s].

Subscripts

c	cold period
h	hot period
n	nth node or cell ($0 \leqslant n \leqslant N$, or $1 \leqslant n \leqslant N$, according to scheme).

Superscripts

0	value at start of period
m	mth period
$'$	derivative
*	modified variable
$-$	(overbar) Laplace transform.

$$t = \frac{\theta - \theta_c}{\theta_h - \theta_c} \tag{4}$$

where θ_h and θ_c are reference hot and cold temperatures. When the hot and cold blows are each at constant temperature, θ_h and θ_c can be chosen to be those constant temperatures, and in that case the hot and cold gas inlet temperatures are 1 and 0, respectively. Further, we define dimensionless time and length:

$$\eta = \frac{hA\tau}{MC} \tag{5}$$

$$\xi = \frac{hAx}{WSL}. \tag{6}$$

In terms of dimensionless quantities, equations (1) and (2) become

$$\frac{\partial T}{\partial \eta} = t - T \tag{7}$$

$$\frac{\partial t}{\partial \xi} = T - t. \tag{8}$$

The cell model of LDR is considered first. This model provides a discretization of equations (7) and (8) based on a physical picture of regenerator operation. The regenerator is divided into N equal cells, in each of which the solid is at a uniform temperature and the gas is well mixed. The cells are labelled by $n = 1, 2, \ldots, N$, and the gas and solid temperatures in cell n are t_n and T_n. Gas flow is in the direction of increasing n (for now). The gas inlet temperature is t_0. The equation for the rate of change of the solid temperature in the nth cell is

$$\frac{dT_n}{d\eta} = t_n - T_n \tag{9}$$

and the gas thermal balance gives

$$t_{(n-1)} - t_n = \lambda(t_n - T_n). \tag{10}$$

Here

$$\lambda = \Lambda / N \tag{11}$$

where $\Lambda = hA/WS$ is the dimensionless (reduced) length of the regenerator. Equations (9) and (10) are the cell model discretization of equations (7) and (8).

LDR have solved equations (9) and (10) using

Laplace transforms with respect to η [4]. The algebraic equations for the Laplace transforms of the gas and solid temperatures, $\bar{t}_n$ and $\bar{T}_n$, are easily solved. The results have the following general form [4]. The outlet gas temperature has Laplace transform (s is the transform variable)

$$\bar{t}_N = \bar{t}_0(s)\bar{\beta}(s,\lambda)+\sum_{i=1}^{N}\lambda s\bar{\alpha}_{(N-i+1)}(s,\lambda)T_i^0 \quad (12)$$

and the solid temperature in the nth cell has Laplace transform

$$\bar{T}_n = \bar{t}_0(s)s\bar{\alpha}_n(s,\lambda)+\sum_{i=1}^{n}\bar{\gamma}_{(n-i)}(s,\lambda)T_i^0. \quad (13)$$

Here T_n^0 is the solid temperature in cell n at time $\eta = 0$. The functions α_n, β and γ_n and their Laplace transforms are given in the Appendix.

The solutions in the time domain are

$$t_N(\eta) = \int_0^\eta t_0(\eta-\zeta)\beta(\zeta,\lambda)\,\mathrm{d}\zeta + \lambda\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}\eta}\alpha_{(N-i+1)}(\eta,\lambda)T_i^0 \quad (14)$$

$$T_n(\eta) = \int_0^\eta t_0(\eta-\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}\alpha_n(\zeta,\lambda)\,\mathrm{d}\zeta+\sum_{i=1}^{n}\gamma_{(n-i)}(\eta,\lambda)T_i^0. \quad (15)$$

The form of the equations (12)–(15) is that which follows from linearity and causality. The outlet gas temperature depends linearly on the inlet gas temperature and on the initial solid temperature in each cell. The solid temperature in the nth cell depends linearly on the inlet gas temperature and the solid temperature in cells m with $m \leqslant n$. This is the structure of, for example, the Hausen heat pole method [1].

The trapezium rule discretization of HW is now considered. The regenerator is divided into $(N+1)$ nodes, with solid temperatures V_n, for $n = 0, 1, \ldots, N$, and with gas temperatures t_n, for $n = 1, 2, \ldots, N$, and t_0 the inlet gas temperature. The discretization of equation (7) is equivalent to equation (9):

$$\frac{\mathrm{d}V_n}{\mathrm{d}\eta} = t_n - V_n. \quad (16)$$

HW use the trapezoidal rule for the discretization of equation (8), which gives

$$t_n - t_{(n+1)} = \frac{\lambda}{2}\{t_{(n+1)} - V_{(n+1)} + t_n - V_n\}. \quad (17)$$

It is possible to take the Laplace transform of equations (16) and (17) and solve for t_N and V_n. The results are equivalent to those of HW [5, 6], who use a different transform to solve the equations. However, a simplification is achieved if instead of the $(N+1)$ nodal solid temperatures V_n, N average solid temperatures are now introduced, defined by

$$T_n = \tfrac{1}{2}(V_n + V_{(n-1)}) \quad (18)$$

for $n = 1, 2, \ldots, N$. Note that the T_n values defined by equation (18) correspond to using temperatures at points half-way between the nodes. Then using a new time variable

$$\eta^* = \frac{2\eta}{2-\lambda} \quad (19)$$

and defining

$$\lambda^* = \frac{2\lambda}{2-\lambda} \quad (20)$$

leads to

$$\bar{t}_N = \bar{t}_0(s^*)\bar{\beta}(s^*,\lambda^*)+\sum_{i=1}^{N}\lambda^* s^*\bar{\alpha}_{(N-i+1)}(s^*,\lambda^*)T_i^0 \quad (21)$$

and

$$\bar{T}_n = \bar{t}_0(s^*)s^*\bar{\alpha}_n(s^*,\lambda^*)+\sum_{i=1}^{n}\bar{\gamma}_{(n-i)}(s^*,\lambda^*)T_i^0 \quad (22)$$

where s^* is the Laplace transform variable corresponding to η^*. Equations (21) and (22) should be compared to equations (12) and (13); they are the same, apart from changes in s and λ. The solutions in the time domain are

$$t_N(\eta) = \int_0^\eta \frac{2}{2-\lambda}t_0(\eta-\zeta)\beta\left(\frac{2\zeta}{2-\lambda},\frac{2\lambda}{2-\lambda}\right)\mathrm{d}\zeta + \lambda\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}\eta}\alpha_{(N-i+1)}\left(\frac{2\eta}{2-\lambda},\frac{2\lambda}{2-\lambda}\right)T_i^0 \quad (23)$$

$$T_n(\eta) = \int_0^\eta t_0(\eta-\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta}\alpha_n\left(\frac{2\zeta}{2-\lambda},\frac{2\lambda}{2-\lambda}\right)\mathrm{d}\zeta + \sum_{i=1}^{n}\lambda_{(n-i)}\left(\frac{2\eta}{2-\lambda},\frac{2\lambda}{2-\lambda}\right)T_i^0. \quad (24)$$

Thus the algebraic structure of LDR and HW is the same; quantities are related by changes in the time scale and changes in parameters.

3. THE CYCLIC STEADY STATE

It is necessary to find the form of these equations in two special cases. The work that follows will be phrased in terms of the cell model, but because of equations (21) and (22), the results will be true, with the appropriate changes, for HW.

The first case is when hot inlet gas with $t_h = 1$ blows into the cell with $n = 1$ for a dimensionless time η_h; the second is when cold inlet gas with $t_c = 0$ blows in the opposite direction, that is into the cell with $n = N$, for a dimensionless time η_c. The results for the solid temperature can be written with remarkable simplicity by introducing a matrix notation. Define matrices

$$\Gamma_{ij} = \gamma_{(i-j)} \tag{25}$$

whose elements are zero when $i < j$, and vectors

$$\mathbf{T} = (T_1, \ldots, T_N)^{\mathrm{T}} \tag{26}$$

$$\alpha = (\alpha_1, \ldots, \alpha_N)^{\mathrm{T}} \tag{27}$$

where the superscript T denotes matrix transpose. If the solid temperature at the start of the hot blow into cell 1 is $\mathbf{T}^0$, then the temperature $\mathbf{T}$ at the end of the hot blow is, from equation (15),

$$\mathbf{T} = \alpha_{\mathrm{h}} + \Gamma_{\mathrm{h}}\mathbf{T}^0 \tag{28}$$

where the subscript h on α and Γ denotes both η_{h} and λ_{h} as arguments. Similarly, if the solid temperature at the start of the cold blow into cell N is $\mathbf{T}^0$, then the temperature $\mathbf{T}$ at the end of the cold blow is, from equation (15), modified to take into account the changed direction of the blow

$$\mathbf{T} = \Gamma_{\mathrm{c}}^{\mathrm{T}}\mathbf{T}^0 \tag{29}$$

where the subscript c denotes both η_{c} and λ_{c}. The matrix Γ_{h} is lower triangular and $\Gamma_{\mathrm{c}}^{\mathrm{T}}$ is upper triangular; equations (28) and (29) display the structure required by linearity and causality. The outlet gas temperature can now be found from equation (14).

This formalism will now be used to set up the equations for the cyclic steady state under conditions such that hot gas with $t_{\mathrm{h}} = 1$ blows into one end of the regenerator for a time η_{h}, and cold gas with $t_{\mathrm{c}} = 0$ blows into the other end of the regenerator for a time η_{c}. Consider a complete cycle consisting of a cold blow followed by a hot blow. Let the solid temperature at the end of the cycle, that is immediately following a hot blow, be $\mathbf{T}^0$. Then after the following cold blow, the solid temperature $\mathbf{T}^{\mathrm{c}}$ is given by equation (29):

$$\mathbf{T}^{\mathrm{c}} = \Gamma_{\mathrm{c}}^{\mathrm{T}}\mathbf{T}^0. \tag{30}$$

After the subsequent hot blow the solid temperature $\mathbf{T}^{\mathrm{h}}$ is given by

$$\mathbf{T}^{\mathrm{h}} = \alpha_{\mathrm{h}} + \Gamma_{\mathrm{h}}\Gamma_{\mathrm{c}}^{\mathrm{T}}\mathbf{T}^0. \tag{31}$$

In the cyclic steady state $\mathbf{T}^{\mathrm{c}} = \mathbf{T}^0$, and these solid temperatures can be found by solving the N inhomogeneous equations represented by equation (31) [5, 6]; the use of the average solid temperature of equation (18) has led to N equations, instead of the $(N+1)$ equations in HW. The physical meaning of equations (30) and (31) has been made manifest by the matrix formalism.

For a particular choice of N, calculations in the two schemes can be related. The thermal effectiveness in HW for dimensionless (reduced) length Λ and dimensionless (reduced) period $\Pi = hAP/MC$, where P is the period, is the same as that in LDR for dimensionless length $2\Lambda/(2-(\Lambda/N))$ and dimensionless period $2\Pi/(2-(\Lambda/N))$.

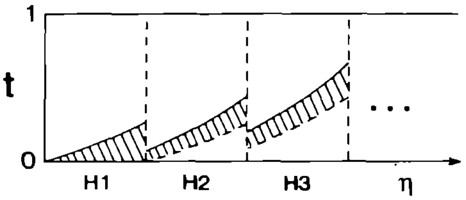

FIG. 1. The outlet gas dimensionless temperature as a function of dimensionless time during the first three hot periods, labelled H1, H2 and H3. The solid line shows the response when the regenerator starts cold, $\mathbf{T} = 0$, the hot inlet gas temperature is $t_{\mathrm{h}} = 1$ and the cold inlet gas temperature is $t_{\mathrm{c}} = 0$. The dashed line is the same response delayed by one cycle.

4. HEAT RECOVERY FOLLOWING A DISTURBANCE

As well as cyclic steady state performance, the transient behaviour of regenerator performance has attracted a great deal of attention [7–11]. An understanding of transient effects is important for the design of regenerator systems where disturbances, such as changes in inlet gas temperatures, occur, so that heat recovery under such conditions can be estimated. Perturbations caused by temporary changes in the inlet gas temperature of a regenerator operating initially in a cyclic steady state with constant hot and cold inlet gas temperatures will be considered. Because of the linearity of the system, it is sufficient to consider the effect of the perturbation on an initially cold regenerator, which, apart from the perturbation, is blown cold from both ends. In order to distinguish between the directions of gas flow, the nomenclature 'hot period' and 'cold period' will be retained.

A special case, when the inlet temperature is increased by unity for one complete hot period, can be dealt with without calculation. The resulting outlet gas temperature during the hot period is found from the difference of two contributions. The first contribution is the response to a permanent unit step increase to the hot inlet gas temperature. The second contribution is the same response delayed by one cycle. By superposition, the difference is the response due to the perturbation. The two contributions are sketched in Fig. 1; their difference is shown by the hatching. The area of the hatching, divided by η_{h}, corresponds to the fraction of heat from the perturbation which is lost from the cold end of the regenerator. The area of the hatching also sums to the area under the cyclic steady state distribution. That area divided by η_{h} is the average dimensionless cold outlet gas temperature, which equals $(1-\varepsilon_{\mathrm{h}})$, where ε_{h} is the hot period thermal effectiveness [1, 2].

The special case was a constant change to the inlet gas temperature for one hot period. When the variation of the inlet gas temperature during a period is important, it is necessary to look in more detail. Using the formalism developed in the previous sections the following theorem will be proved.

Theorem. Consider an impulse perturbation to the inlet gas temperature, $\delta(\eta-\eta_0)$, which occurs at dimensionless time η_0 from the start of a hot period. The fraction of the heat from the impulse which is lost from the cold end of the regenerator is equal to the dimensionless temperature of the gas leaving the cold end of the regenerator, in the cyclic steady state, at time $(\eta_h-\eta_0)$ from the start of a hot period.

Corollary. The heat lost from a general perturbation to the hot inlet gas temperature can then be found from the cold outlet cyclic steady state temperature distribution by convolution.

The calculations are done in two parts. In the first part, the effects of the impulse perturbation are analysed. As was stated earlier, because the system is linear, this can be done by imposing the perturbation on a regenerator *which is initially cold* and which is blown cold from both ends. The heat lost from one end of the regenerator during each subsequent cycle is then found. In the second part of the calculation, the regenerator starts cold and an ordinary development to cyclic steady state is followed.

4.1. *Impulse response*

To repeat: although, apart from the impulse, the regenerator is blown cold from both ends, the nomenclature 'hot period' and 'cold period' will be retained. There is a perturbation $\delta(\eta-\eta_0)$ to the inlet temperature in one hot period.

During this hot period, the outlet gas temperature can be found from equation (14) with $t_0 = \delta(\eta-\eta_0)$:

$$t_N(\eta) = \beta(\eta-\eta_0, \lambda_h) \quad \text{for } \eta > \eta_0$$
$$t_N(\eta) = 0 \quad \text{for } \eta < \eta_0. \tag{32}$$

Then the fraction of heat lost in the gas leaving the cold end of the regenerator is

$$Q_1 = \int_0^{\eta_h} t_N(\eta)\,\mathrm{d}\eta = \int_0^{\eta_h-\eta_0} \beta(\zeta, \lambda_h)\,\mathrm{d}\zeta$$
$$\equiv b(\eta_h-\eta_0, \lambda_h) \tag{33}$$

defining the function b. The solid temperature at the end of this hot period is, from equation (15)

$$\mathbf{T}^{1,h} = \alpha'(\eta_h-\eta_0, \lambda_h) \tag{34}$$

where the prime denotes differentiation with respect to the first argument. The calculation now follows the process period by period and sums up the heat lost from the cold end of the regenerator during each hot period.

The solid temperature at the end of the first cold period (which follows the hot period that has just been considered) is, from equation (29)

$$\mathbf{T}^{1,c} = \Gamma_c^T \alpha'(\eta_h-\eta_0, \lambda_h). \tag{35}$$

Apart from the impulse, the gas is blown cold from both ends of the regenerator. The solid temperature at the end of the $(m-1)$th cold period, with $m \geqslant 2$, is

$$\mathbf{T}^{(m-1),c} = [\Gamma_c^T\Gamma_h]^{(m-2)}\Gamma_c^T\alpha'(\eta_h-\eta_0, \lambda_h). \tag{36}$$

From this, the gas outlet temperature in the mth hot period can be found from equation (14); its integral is the fraction of heat lost during the mth hot period:

$$Q_m = \lambda_h \sum_{i=1}^{N} \alpha_{(N-i+1)}(\eta_h, \lambda_h) \times \{[\Gamma_c^T\Gamma_h]^{m-2}\Gamma_c^T\alpha'(\eta_h-\eta_0, \lambda_h)\}_i. \tag{37}$$

4.2. *Development of the cyclic steady state*

When the regenerator is blown alternately with hot gas, $t_h = 1$, from one end, and cold gas, $t_c = 0$, from the other, it eventually reaches a steady state independent of the initial temperature distribution. For the present purpose it is useful to start the regenerator cold and start the hot blow at the beginning of a period.

The outlet gas temperature at time $(\eta_h-\eta_0)$ from the start of the first hot period can be found from equations (14) and (32):

$$t_N^{(1,h)}(\eta_h-\eta_0) = b(\eta_h-\eta_0, \lambda_h) \tag{38}$$

which equals Q_1. Now the evolution of the system will be followed in the same way as before, and it will be found that the difference in outlet temperatures at a time $(\eta_h-\eta_0)$ from the start of the mth and $(m-1)$th hot periods is Q_m, as given by equation (37).

From equation (28) the solid temperature at the end of the first hot period is

$$\mathbf{T}^{1,h} = \alpha_h \tag{39}$$

and from equation (24) the solid temperature at the end of the first cold period is

$$\mathbf{T}^{1,c} = \Gamma_c^T\alpha_h. \tag{40}$$

The solid temperatures at the end of the $(m-1)$th and $(m-2)$th cold periods $(m \geqslant 2)$ are related by

$$\mathbf{T}^{(m-1),c} = \mathbf{T}^{(m-2),c} + (\Gamma_c^T\Gamma_h)^{(m-2)}\Gamma_c^T\alpha_h \tag{41}$$

where $\mathbf{T}^{0,c} = \mathbf{0}$. The difference between outlet gas temperatures during the mth and $(m-1)$th hot periods can be found using equation (14). Evaluating this difference at time $(\eta_h-\eta_0)$ from the start of the hot period gives

$$t_N^{(m,h)}(\eta_h-\eta_0) - t_N^{(m-1,h)}(\eta_h-\eta_0)$$
$$= \lambda_h \sum_{i=1}^{N} \alpha'_{(N-i+1)}(\eta_h-\eta_0, \lambda_h)[(\Gamma_c^T\Gamma_h)^{(m-2)}\Gamma_c^T\alpha_h]. \tag{42}$$

It will now be proved that the right hand side of equation (42) equals Q_m given by equation (37). This can be done by introducing new matrices

$$G_{ij} = \Gamma_{(N-i+1)j}. \tag{43}$$

The G values have three useful properties:

(i) they are symmetric and thus equal to their own transpose;

(ii) $\Gamma_c^T\Gamma_h = G_cG_h$; (44)

(iii) for a vector $\mathbf{x}$

$$(\Gamma_c^T \mathbf{x})_i = \sum_{i=1}^{N} G_{ij} x_{(N-i+1)}. \qquad (45)$$

The required result follows immediately from these properties.

The cyclic steady state outlet gas temperature during a hot period, t_N^h, can, in the same way as for the special case, as indicated in Fig. 1, be built up as the sum of increments from period to period. Evaluating this temperature at time $(\eta_h - \eta_0)$ from the start of a hot period gives

$$t_N^h(\eta_h - \eta_0) = t_N^{(1,h)}(\eta_h - \eta_0) + \sum_{m=2}^{\infty} [t_N^{(m,h)}(\eta_h - \eta_0) - t_N^{(m-1,h)}(\eta_h - \eta_0)]. \qquad (46)$$

It has been shown that the mth term on the right hand side of equation (46) is equal to Q_m, so then

$$t_N^h(\eta_h - \eta_0) = \sum_{m=1}^{\infty} Q_m = Q_{tot} \qquad (47)$$

where Q_{tot} is the total fraction of heat from the impulse lost from the cold end of the regenerator. Thus the theorem is proved.

Note that this result is exact for both LDR and HW, and for an arbitrary number of cells or nodes. So it will be true as that number tends to infinity, that is for the solution of equations (7) and (8). A calculation using a discretization of the governing differential equations has led to an exact result for the solutions of the differential equations themselves.

4.3. *Examples*

A general perturbation in the hot inlet gas temperature can be split into contributions over separate hot periods, which can be considered separately. A perturbation in the inlet gas temperature occurring in a particular hot period, $g(\eta)$, can be viewed as a linear combination of impulses:

$$g(\eta) = \int_0^{\eta_h} \delta(\eta - \zeta) g(\zeta)\, d\zeta. \qquad (48)$$

The fraction of heat from the perturbation which is lost from the cold end of the regenerator, Q_{tot}, can then be found from a weighted average of cyclic steady state outlet gas temperatures:

$$Q_{tot} = \int_0^{\eta_h} t_N^h(\eta_h - \zeta) g(\zeta)\, d\zeta \Big/ \int_0^{\eta_h} g(\zeta)\, d\zeta. \qquad (49)$$

As a first example, this result will be applied to the special case described earlier, of a constant perturbation to the hot inlet gas temperature lasting for one cycle. In that case $g(\eta) = 1$, and equation (49) gives Q_{tot} to be the average outlet gas temperature during the hot period, as required by the general argument for this perturbation, given at the beginning of Section 4.

As a second example, consider a regenerator with $\Lambda_h = \Lambda_c = 9$ and $\eta_h = \eta_c = 3.5$, numbers typical of a regenerative burner (see, e.g. refs. [11, 12]). The cyclic steady state outlet gas temperature during a hot period is shown in Fig. 2. The thermal effectiveness is $\varepsilon = \varepsilon_h = \varepsilon_c = 0.802$.

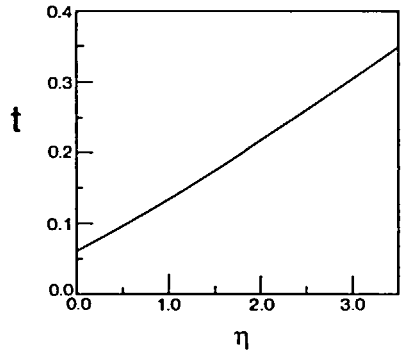

FIG. 2. The outlet gas dimensionless temperature as a function of dimensionless time in the cyclic steady state during a hot period. $\Lambda_h = \Lambda_c = 9$, $\eta_h = \eta_c = 3.5$.

Table 1. Dependence of Δt on Λ and Π from ref. [14]

Λ	Π	Δt
50	1	0.002
40	20	0.23
5	5	0.61

For an impulse perturbation occurring at the start of the hot period, $g(\eta) = \delta(\eta)$, the fraction of heat lost from the cold end of the regenerator is given by the outlet gas temperature at time $\eta = \eta_h$, i.e. $Q_{tot} = 0.347$. Similarly, for an impulse perturbation occurring at the end of a hot period, $g(\eta) = \delta(\eta - \eta_h)$, $Q_{tot} = 0.061$. The fraction of heat lost from a general perturbation will lie between these two limits. In particular, if the perturbation is of constant magnitude, $Q_{tot} = 1 - \varepsilon = 0.198$.

The effect can be appreciable when the outlet gas temperature varies significantly during a period. For large dimensionless length Λ and small dimensionless period Π, there will be little variation in the outlet gas temperature during a period; for small Λ and large Π, there will be a large variation in the outlet gas temperature during a period. A detailed study has been carried out by Heggs *et al.* [13]. Sample results on the variation in outlet gas temperature during a period Δt are shown in Table 1.

5. DISCUSSION

Laplace transform techniques have been used to compare two models of linear regenerator operation. It has been shown that the trapezium rule model of Hill and Willmott and the cell model of Lai, Dudukovic and Ramachandran have the same algebraic structure. This structure follows from linearity and causality, and corresponds to that of the Hausen heat pole method. The numerical results of HW are related to those of LDR by a change of the time scale and changes of parameters.

The formalism developed has been used to elucidate the physical interpretation of the equations for cyclic steady state temperature distribution, and to study heat recovery following a perturbation in the hot inlet gas temperature. It has been shown that the fraction of heat lost from the cold end of the regenerator during hot periods is given by a weighted average of the cyclic steady state outlet gas dimensionless temperature distribution. This fraction is bounded by the cyclic steady state outlet gas dimensionless temperature at the start and end of a hot period. Equivalent results hold for perturbations in the cold inlet gas temperature.

Further consequences of the algebraic structure of discretizations of equations (7) and (8) are reported by Willmott *et al.* [14]. The application of these techniques to transient regenerator behaviour will be the subject of future work.

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APPENDIX

The coefficient functions in equations (12)–(15) are

$$\bar{\alpha}_n(s,\lambda) = \frac{(s+1)^{n-1}}{[(1+\lambda)s+1]^n s}$$

$$\bar{\beta}(s,\lambda) = \left(\frac{s+1}{(1+\lambda)s+1}\right)^N$$

$$\bar{\gamma}_n(s,\lambda) = \frac{1+\lambda}{(1+\lambda)s+1} \qquad n=0$$

$$= \frac{\lambda(s+1)^{n-1}}{[(1+\lambda)s+1]^{n+1}} \qquad n \geqslant 1$$

$$\alpha_n(\eta,\lambda) = \frac{1}{(1+\lambda)^n}\sum_{k=0}^{n-1}\left(\frac{\lambda}{1+\lambda}\right)^k \binom{n-1}{k}\frac{1}{k!}\int_0^\eta \sigma^k\, e^{-\sigma/(1+\lambda)}\, d\sigma$$

$$\beta(\eta,\lambda) = \frac{1}{(1+\lambda)^N}\left[\delta(\eta) + \sum_{k=1}^{N}\left(\frac{\lambda}{1+\lambda}\right)^k \binom{N}{k} \times \frac{1}{(k-1)!}\eta^{k-1}\, e^{-\eta/(1+\lambda)}\right]$$

$$\gamma_n(\eta,\lambda) = e^{-\eta/(1+\lambda)} \qquad n=0$$

$$= \frac{\lambda}{(1+\lambda)^{n+1}}\sum_{k=0}^{n-1}\left(\frac{\lambda}{1+\lambda}\right)^k \binom{n-1}{k} \times \frac{1}{(k+1)!}\eta^{k+1}\, e^{-\eta/(1+\lambda)} \qquad n \geqslant 1.$$